Vacation Work: Maths & Mechanics

The Oxford Physics course contains a lot of maths and it is important to ensure that you are as far as possible up to speed before arriving in Oxford. This is particularly important if you have not studied Further Maths or equivalent. This document gives some advice on preparation before arrival, and a set of problems which you should attempt over the summer.

Maths Textbooks

The recommended maths text for the course is *Mathematical Methods for Physics and Engineering* by K. F. Riley, M. P. Hobson and S. J. Bence (3rd edition, Cambridge University Press). This is a very large book, but will cover all the maths you need for the compulsory parts of the physics course. You are advised, if possible, to obtain a copy before coming up to Oxford.

You may initially find this text very tough. A useful transition text is *Further Mathematics for the Physical Sciences* by Tinker and Lambourne, which is divided into a number of modules with self assessment tests so that you can work out which chapters you need to read.

Introductory Problems

We will assume that you are familiar with Chapter 1 of Riley, Hobson and Bence on arrival. This chapter covers the solution of polynomials, trigonometric identities, coordinate geometry, partial fractions, the binomial expansion, and methods of proof. The problems listed below are adapted from the end of this chapter. You may find these problems quite tricky at first, but with thought they should be possible. If you find them easy then look at some other problems in Chapter 1.

Calculus

Although the Oxford Physics course in principle covers calculus from scratch, in practice elementary techniques are covered *very* quickly, and you should

ensure that you are familiar with these before arrival. This includes differentiation and integration of polynomials, trigonometric functions, and exponential and logarithmic functions; differentiation of products and quotients and use of the chain rule; implicit differentiation; integration by parts and by substitution.

Mechanics

The first "physics" subject you will study at Oxford is mechanics. Unlike school courses the Oxford course quickly begins to make use of a significant amount of maths, and many students find it hard to *combine* maths with physics. This involves taking a problem posed in ordinary language, setting it up in precise mathematical terms, using mathematical techniques to find a solution, and then translating this back into ordinary terms. This process may not be very familiar to you now but is going to be very important throughout your course.

The mechanics problems provided can mostly be solved using techniques familiar from A-level, but it is important to begin by adopting a proper approach straight away: the way in which you solve these problems and set out your answers is more important than the answers themselves! In particular you should avoid at all costs simply plugging numbers into standard formulae using a calculator: you should solve each problem *in the general case* before dealing with the numbers given.

A. Introductory Problems

1. A polynomial equation can be written in terms of coefficients

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

or in terms of roots

$$f(x) = a_n(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) = 0.$$

(a) By expanding these expressions show that

$$\sum_{k=1}^{n} \alpha_k = -\frac{a_{n-1}}{a_n} \quad \text{and} \quad \prod_{k=1}^{n} \alpha_k = (-1)^n \frac{a_0}{a_n}$$

where Π indicates a product of terms just as Σ indicates a sum.

(b) Consider the polynomial $g(x) = 4x^3 + 3x^2 - 6x - 1$. Show that $x = 1$ is a root, and find the sum and product of the other two roots.

2. Use double angle formulae to find an expression for $\cos(4\theta)$ in terms of $\sin(\theta)$; hence prove that $s = \sin(\pi/8)$ is one of the four roots of

$$8s^4 - 8s^2 + 1 = 0$$

and use your knowledge of the sin function to show that

$$\sin(\pi/8) = \sqrt{\frac{2 - \sqrt{2}}{4}}.$$

Find an expression for $\cos(\pi/8)$ and show that $\tan(\pi/8) = \sqrt{2} - 1$. (Hint: first find the "obvious" form for $\tan(\pi/8)$ and then simplify this using surds.)

3. Show that

$$a\sin\theta + b\cos\theta = K\sin(\theta + \phi)$$

with

$$K^2 = a^2 + b^2 \quad \text{and} \quad \phi = \arctan(b/a).$$

(Hint: it is much easier to show this "the other way round" by expanding $K\sin(\theta + \phi)$ using an angle sum formula.)

4. Show that the equation

$$f(x, y) = x^2 + y^2 + 6x + 8y = 0$$

represents a circle, and find its centre and radius.

5. Express the following as partial fractions:

$$\text{(a)} \quad \frac{2x+1}{x^2+3x-10} \quad \text{(b)} \quad \frac{4}{x^2-3x} \quad \text{(c)} \quad \frac{x^2+x-1}{x^2+x-2}$$

$$\text{(d)} \quad \frac{2x}{(x+1)(x-1)^2} \quad \text{(e)} \quad \frac{2+4x}{(x+2)(x^2+2)}$$

(In case (c) you should start by dividing to find the quotient and remainder; in cases (d) and (e) you will need three constants.)

6. Use a binomial expansion to evaluate (a) $(1+x)^5$, (b) $1/(1+x)$, and (c) $1/\sqrt{1+x}$ up to second order. Use the last result to find a value of $1/\sqrt{4.2}$ to three decimal places.

7. Prove by induction that

   (a) $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

   (b) $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

8. The sum of a geometric progression can be written as

$$S_n = 1 + r + r^2 + \cdots + r^n$$

Evaluate rS_n and use this to show that $S_n = (1-r^{n+1})/(1-r)$.

B. Calculus

Differentiation

1. Differentiate each of the following functions:

   (a) $f(x) = x^2 \sin(x) + \log_e(x)$

   (b) $f(x) = 1/\sin(x)$

   (c) $f(x) = 2^x$

2. Find the first two derivatives of the following functions:

   (a) $F(x) = 3\sin x + 4\cos x$

   (b) $y = \log_e x \quad x > 0$

3. Let $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, where a is a constant; find $\frac{dy}{dx}$ in terms of θ. Then find d^2y/dx^2 in terms of θ.
   (Hint: let $f(\theta)$ stand for your expression for $\frac{dy}{dx}$ in terms of θ; then $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{d}{dx}f(\theta) = \frac{df}{d\theta}\frac{d\theta}{dx}$.)

Stationary points and graph sketching

4. Under certain circumstances, when sound travels from one medium to another, the fraction of the incident energy that is transmitted across the interface is given by

   $$E(r) = 4r/(1+r)^2$$

   where r is the ratio of the acoustic resistances of the two media. Find any stationary points of $E(r)$, and classify them as local maxima, minima, or points of inflection. For what value of r is $\mathrm{d}^2E/\mathrm{d}r^2 = 0$? Sketch the graph of $E(r)$ for $r > 0$, marking the special values of r you have found.

Hyperbolic functions

5. The hyperbolic functions are defined in terms of the exponential function e^x by

   $$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

   Sketch each of these functions against x. Do not use a calculator!

6. Find the first derivatives of $\sinh(x)$ and $\tanh(x)$ with respect to x, expressing your answers in terms of $\cosh(x)$.

Integration

7. Evaluate the following indefinite integrals: (a) $\int(1 + 2x + 3x^2)\,\mathrm{d}x$, (b) $\int[\sin(2x) - \cos(3x)]\,\mathrm{d}x$, (c) $\int(e^t + \frac{1}{t^2})\,\mathrm{d}t$, (d) $\int \mathrm{d}w$.

8. Evaluate the following definite integrals:

   (a) $\int_{-1/4}^{1/4} \cos(2\pi x)\,\mathrm{d}x$

   (b) $\int_0^3 (2t - 1)^2\,\mathrm{d}t$

   (c) $\int_1^2 \frac{(1+e^t)^2}{e^t}\,\mathrm{d}t$

   (d) $\int_4^9 \sqrt{x}(x - \frac{1}{x})\,\mathrm{d}x$

   (e) $\int_{-1}^1 x^3\,\mathrm{d}x$; explain your answer with a sketch.

9. Find the indefinite integral $\int x^2 e^{-x}\,\mathrm{d}x$.

10. Find the indefinite integral $\int \sin x(1 + \cos x)^4\,\mathrm{d}x$.

11. Find the area of the region bounded by the graph of the function $y = x^2 + 2$ and the line $y = 5 - 2x$.

12. Although integrals are frequently thought of as areas under curves, a more general interpretation is possible and useful.

    (a) Justify the standard formula for the volume of revolution obtained by rotating the curve $y = f(x)$ for the interval $a < x < b$ by 2π around the x axis. (Start by slicing the volume up into infinitely thin coaxial discs and sum the volumes of these.) Find the volume of Torricelli's trumpet, defined by $y = 1/x$ for $1 < x < \infty$.

    (b) Consider an elastic spring of natural length l m which has a tension $k\,x^2$ N when it is stretched by x m. Find the work done when this spring is stretched to twice its natural length. (Note: this spring does not obey Hooke's law and so standard results do not apply; start by finding the work done when the spring is stretched by an infinitesimal amount $\mathrm{d}x$.)

C. Mechanics

Motion in one dimension

1. A car accelerates uniformly from rest to 80 km per hour in 10 s. How far has the car travelled?

2. A stone falls from rest with an acceleration of $9.8\,\mathrm{ms}^{-2}$. How fast is it moving after it has fallen through 2 m?

3. A car is travelling at an initial velocity of $6\,\mathrm{ms}^{-1}$. It then accelerates at $3\,\mathrm{ms}^{-2}$ over a distance of 20 m. What is its final velocity?

Work and energy

4. The brakes on a car of mass 1000 kg travelling at a speed of $15\,\mathrm{ms}^{-1}$ are suddenly applied so that the car skids to a stop in a distance of 30 m. Use energy considerations to determine the magnitude of the total frictional force acting on the tyres, assuming it to be constant throughout the braking process. What is the car's speed after the first 15 m of this skid?

5. The gravitational potential energy for a mass m at a distance $R + h$ from the centre of the earth (where R is the radius of the earth) is $-GMm/(R + h)$ where G is Newton's gravitational constant and M is the mass of the earth. If $h \ll R$ show that this is approximately equal to a constant (independent of h) plus mgh, where $g = GM/R^2$. [Hint: write $R + h = R(1 + h/R)$ and expand $(1 + h/R)^{-1}$ by the binomial theorem.]

6. Show that the minimum speed with which a body can be projected from the surface of the earth to enable it to just escape from the earth's gravity (and reach "infinity" with zero speed) is given by $v_{escape} = \sqrt{2GM/R}$, where G, M and R are defined as in the previous question. If a body is projected vertically with speed $\frac{1}{2}v_{escape}$, how high will it get? (Give the answer in terms of R and neglect air resistance throughout this question.)

Simple harmonic motion

7. The position of a particle as a function of time is given by $x(t) = A\sin(\omega t + \phi)$, where A, ω and ϕ are constants. Obtain similar formulae for (a) the velocity, (b) the acceleration of the particle. If $\phi = \pi/6$, find in terms of A the value of x at $t = \pi/6\omega$. What is the value of x (in terms of A) when the acceleration is greatest in magnitude?

8. A particle of mass m moves in one dimension under the action of a force given by $-kx$ where x is the displacement of the body at time t, and k is a positive constant. Using $F = ma$ write down a differential equation for x, and verify that its solution is $x = A\cos(\omega t + \phi)$, where $\omega^2 = k/m$. If the body starts from rest at the point $x = x_0$ at time $t = 0$, find an expression for x at later times.